SUBSTITUTION.

[*Off-printed from* MIND: *a Quarterly Review of Psychology and Philosophy. Vol. XLVII., N.S., No.* 188.]

SUBSTITUTION in symbolic logic is used as a method by which new true propositions can be derived from known or given truths. It is the function of a principle of substitution to indicate which modes of substitution are legitimate, which illegitimate, within a given system. The *Principia Mathematica* states no law of substitution, but it is possible to derive one by setting up the following table of modes of substitution accepted by the *Principia*.[1]

	Given.	*Comments.*	*Accepted Substitutions.*	*Restrictions.*
I.	p, q	real variables, each representing the same range of values.	$p/q, q/p$ as: $\vdash : p . \supset . q \vee p$ $p/q \vdash : p . \supset . p \vee p$ $q/p \vdash : q . \supset . q \vee q$	Law of completeness —the substitution must be made in every occurrence of the variable in the given proposition.
II.	$p \equiv q$	ordinary equivalent propositions.	$p/q, q/p$ as: $\vdash : p . \supset . p \vee q$ $p/q \vdash : p . \supset . p \vee p$ $q/p \vdash : q . \supset . q \vee q$	Law of completeness —authors fail to recognise that substitution of equivalents need not be made throughout.[2]
III.	$p \supset q = -p \vee q$ def.	$-p \vee q$ is known and $p \supset q$ is defined in terms of it.	$p \supset q/-p \vee q$ as: $\vdash : q . \supset . -p \vee q$ $p \supset q/-p \vee q$ $\vdash : q . \supset . p \supset q$	$-p \vee q/p \supset q$ ruled out for methodological reasons—that is, definition proceeds in a linear direction to give P.M. the appearance of a deductive system.[3]

[1] This table has been suggested by Dr. Paul Weiss. "Given" does not mean "Given that" in any case, but simply *denotes* the variables or expression as that with which we are working. Thus "Given $p \equiv q$" does *not* mean either "$\vdash . p \equiv q$," or "given that $p \equiv q$."

[2] Substitution of equivalents need not be made throughout because such a substitution can never make an assertible of a non-assertible. *Cf.* P. Weiss, *Two Valued Logic*, Erkenntnis II, Band Heft 4, 1931, p. 249 ff.

[3] Although the propositions of the *Principia* can be deduced from various sets of primitives, and in several ways, definition is introduced to indicate the one particular mode of derivation chosen by the authors.

	Given.	*Comments.*	*Accepted Substitutions.*	*Restrictions.*
IV.	$p \supset q, p$	p is a variable and $p \supset q$ an elementary function of propositions.	$p \supset q/p$ as: $\vdash : pq . \supset . p \supset q$ $p \supset q/p$ $\vdash :: p \supset q : q . : \supset .: p \supset q . \supset . q$	Law of completeness (see I above). Rule out $p/p \supset q$ because it would make possible the derivation of assertibles from non-assertibles.
V.	$p, pq \vee p - q$	$pq \vee p - q$ is the necessary proposition $q \vee - q$ conjoined with the unassertible element p.	$pq \vee p - q/p$ as: $\vdash : p . \supset . q \vee p$ $pq \vee p - q/p$ $\vdash :. pq \vee p - q : \supset : q . \vee . pq \vee p - q$	The authors regard this as a form of IV, though by so doing they falsify their own distinction between assertible elements ($q \vee - q$) and non-assertible elements ($p \vee q$). Thus, by analogy they rule out $p/pq \vee p - q$, which they should not do. This is better regarded as a form of II, p and $pq \vee p - q$ being equivalents.

There are two difficulties in the *Principia's* theory of substitution as it is expressed in the table above. First, following the *Principia*, substitution I, we get:

$$1.\ \vdash : p . \supset . q \vee p.$$
$$2.\ p/q \vdash : p . \supset . p \vee p.$$

The *Principia* regards 2 as a special case of 1, thus belying its own basic principle that all elementary propositions stand on one level.[1] For the fact that it is impossible to derive proposition 1 from proposition 2 by any accepted principle of substitution indicates that the p of proposition 2 is more general than either the p or q of 1, and that it subsumes them both.[2] The two propositions are analogous to the two linguistic expressions, "two apples and four oranges," and "two pieces of fruit and four pieces of fruit." In the second statement both apples and oranges as such are lost in a more inclusive term. Similarly, in 2, both the p and the q of 1 are lost in a less definitive p. However, if 2 is not derived from 1, there is no other derivation of it possible within the *Principia* as it stands (because it cannot be

[1] *Principia Mathematica*; Introduction, 2nd edition, p. 6 f. (Level here has no reference to the typal hierarchy, but only to the assumed equivalence of all elementary propositions; equivalence in the sense that the variable "p" may stand for any of them.)

[2] This could be interpreted as meaning that the p's in 1 and 2 are the same, but that the q has been lost. But this would mean that the p in 1 and 2 was more general than the q, and would imply a hierarchy of propositions such as the *Principia Mathematica* explicitly denies.

derived from *1 . 2, and all the other primitives have two variables, and thus are just another case of 1) and, since the proposition is an important one, it must be introduced as a primitive. However, it seems desirable to find some less radical alteration of the *Principia* which would permit the derivation of 2 without adding to the list of primitives.

The second difficulty lies in the fact that there is no law of substitution stated in the *Principia*. If such a statement could be made, it would result in the internal clarification of the *Principia* and in a better understanding of the relation of the *Principia* to other specific logical systems, and to the general nexus of meaning to which all such systems belong.

The first difficulty leads to certain restrictions on substitution as a method of derivation within the *Principia*; the second demands that these restrictions be formulated in a law which could be given a place among the postulates, or laws of operation, of the *Principia*. The following restrictions, in addition to those imposed by the authors of the *Principia*, seem to be indicated.

(1) Substitutions under mode I are to be allowed only when the variable being substituted does not already occur in the proposition in which the substitution is being made. Substitution of the type p/q is permissible only in three cases: where $p \equiv q$; where $p = q$df; and where p does not already occur, as in the proposition, $q \supset q$. Such substitutions of p/q where q does already appear as in I above are here ruled out on the ground that they involve a loss of structure. Eaton says, "Symbols originally taken as distinct in meaning must be construed as distinct throughout. None of the transformations of the system can result in the use of distinct symbols as the same or equivalent. For this would violate the principle of contradiction, which asserts that any possible value of a negative, that is, any symbol distinct from a given symbol, is always distinct from this symbol."[1] It is true that p and q, representing elementary propositions, can each have all the values the other may have, when they are standing alone. Thus the propositions $p \vee p . \supset . p$ and $q \vee q . \supset . q$ are in every sense identical. But the importance of the phrase "standing alone" is too often neglected in the *Principia*. For if that phrase is taken seriously, it is clear that p cannot be substituted for q in every context, without violating the law of contradiction. No matter what may be true of p and q standing alone, their appearance together in a single proposition means that each is a case of the denial of the other.[2] My assumption, and I think that of common sense, is that, although there is no value of one which is not also a possible value for the other, both of them never possesses the same value at the same time. Whatever p may be in a given proposition, q is other than p, or a case of $-p$. Thus

[1] Eaton, *Symbolism and Truth*, p. 226.

[2] This expression is used to mean merely "each is a different proposition from the other."

consider the proposition, $p \,.\supset .\, p \vee q$ and suppose that x and y constitute the range of values possible to p and q alike. We get, then, the following set of possible combinations of value in the proposition.

	p	q
1.	x	x
2.	x	y
3.	y	x
4.	y	y

Now, if we assume that when p and q appear in the same proposition, p is different from q, we must eliminate cases 1 and 4 above as not being instances of the propositions p and q in this context. The *Principia*, however, by assuming that such a proposition as 2 can be derived from proposition 1 by substitution, implicitly accepts these cases as possible instances of the propositions in question. This assumption denies that there need be any difference between p and q, and the denial of difference makes substitution impossible.[1] Thus substitutions of form I are legitimate only when p and q are recognised as variables which, standing alone, have the same range of values, but standing together are each instances of the denial of the other—that is, never have the same value at the same time. p can be substituted for q only when it does not already appear as a variable in the proposition in question.

(2) The second limitation is to be imposed upon substitutions of type IV. Substitution of $p/p \supset q$ must be rejected, as the authors of the *Principia* recognise, because it can involve a considerable loss of structure. But they fail to recognise that substitutions of $p \supset q/p$, if made in propositions in which q already occurs as a variable, entail a similar loss of structure. Like the form of substitution eliminated under (1) above, this too involves an incomplete recognition of the difference between p and q.[2] Substitutions of the type $p \supset q/p$ are legitimate where q does not already appear as a variable, under the limitations imposed by the law of completeness. This proviso is necessary because without it we should be able to assert non-assertibles.[3] The law of completeness is but another way

[1] Bradley, *Principles of Logic*, p. 375.

[2] The substitution of $p \supset q/q$ in $p \vee q$ would make a contingent into a necessary truth, $p \,.\vee .\, p \supset q$. Although this error cannot be made in the *Principia*, because they start with necessary truths, it is one which their scheme of substitution makes possible.

[3] Thus consider the substitution $p \vee - p/p$ in $p \supset p$. If this is not made throughout we get $p \vee - p \,.\supset .\, p$, and, since $\vdash : p \vee - p$, $\vdash p$. But p is not an assertible. The law of completeness avoids this, — thus:

$$\begin{array}{ll} & p \supset p \\ p \vee - p/p & p \vee - p \supset p \vee - p \\ & \vdash : p \vee - p \\ & \therefore \vdash : p \vee - p. \end{array}$$

Note that this differs from case V because $p \not\equiv p \vee - p$.

of saying that the *Principia* is interested only in assertible propositions.

The first difficulty is to be settled by formulating the principles of substitution proper to the *Principia* in a law which can be stated explicitly in the *Principia*. This law would read: (1) all substitution of equivalents [1] and all substitution of definiendum for definiens is legitimate without restriction in all or some occurrences of the variable; (2) all substitution of q/p or $p \supset q/p$ is legitimate only when q does not already appear as a variable in the proposition and, even then, only under the restrictions of the law of completeness; [2] (3) all substitution of $p/p \supset q$, involving, as it does, loss of structure, is illegitimate.

Two questions remain. First, do these restrictions permit the derivation of all the propositions of the *Principia*, and second, do they answer the objection raised to the present unrestricted use of substitution? In answer to the first question, examination of the *Principia* reveals that every proposition stated in terms of two or three variables can be derived by means of the restricted forms of substitution.[3] Propositions in one variable, however, are a different problem. Among the primitive propositions there is one in one variable, *1·2, which states $p \vee p . \supset . p$. The proposition, $p . \supset . p \vee p$ cannot be derived directly from this, nor can it be derived from *Principia* *1·3, if the restrictions of substitution here suggested are accepted.[4] In itself, this proposition seems quite legitimate, and it is certainly important, since from it are derived the laws of excluded middle and identity, the principle of *reductio ad absurdum* and its complement, and many others. We must, therefore, either add the proposition $p . \supset . p \vee p$ to our list of primitives, or add a definition which will permit its derivation from the present list of primitives. The second alternative seems preferable because it involves the least change within the system, and because it takes care, in one step, of the derivation of all the propositions in one variable. The necessary definition, as suggested and formulated for

[1] Substitutions of form V in table to be regarded as a form of II, or substitution of equivalents.

[2] The law of completeness is thus not a separate law, but a principle restricting certain forms of substitution.

[3] Those in three variables would be taken care of by (2) of the rules of substitution, which makes possible the introduction of a new variable at any point. My objection to Dr. Weiss's suggestion in *Two Valued Logic* that all derivation be instantaneous, and accomplished by the substitution of equivalents, rests on the fact that, in his system, the introduction of a third variable would require an entirely new table of logical constants. This implies that his method lacks the generality which a logical system should have (*op. cit.*, p. 251 f.).

[4] In the *Principia* this proposition, *2·01, is derived by substituting p/q in *1·3, a substitution here ruled out.

me by Dr. Weiss, should be added to the *Principia* as *1·02. It should read: *1·02 $p = p \vee p = p$. pdf.[1]

In answer to the second question, the restrictions here suggested do answer the objections raised earlier. First, the hierarchy of propositions implied (but explicitly denied) in the *Principia's* method of substitution is broken down. Second, the explicit statement in the *Principia* of this restricted principle of substitution would clarify the *Principia* itself, and shed light on the relation of the *Principia* to other systems. On general epistemological grounds it seems justifiable to assume that all possible forms of substitution constitute legitimate means of derivation in some system. The fact that all are not legitimate within the *Principia* shows that the *Principia* does not pretend to cover every possible form of meaningful expression, but only a restricted realm within the realm of meanings. Further, a system, if such there be, which accepted the substitutions which the *Principia* rejects, and rejected those which the *Principia* accepts, would bear an immediately defined relationship to the *Principia*. Finally, the statement of this restricted law of substitution among the primitive propositions of the *Principia* eliminates the symbol "⊦." A statement of the kind of substitution allowed tells us that all propositions so derived will have the same truth value as the primitive propositions, which is all that "⊦" means. Further, it is unnecessary to "assert" the primitives, since either they, like their derivates are true, or the authors are to be convicted of error.[2] Thus, a statement of the principle of substitution, and rigid adherence to that statement, guarantees internal consistency. The derived propositions are no more true than the primitives—but neither are they less true. The explicit statement of substitution as a mode of operation within the *Principia* shows that all the primitives and derivates of the system have the same truth value, and thus eliminates the need for a distinction within the system between assertibles and non-assertibles such as "⊦" introduces.[3]

Bryn Mawr College. MARJORIE GOLDWASSER WYLER.

[1] *2·07 $\vdash : p \,.\, \supset .\, p \vee p$

Dem.

*1·2 $\vdash : p \vee p \,.\, \supset .\, p$

[*1·01] $\vdash : -[p \vee p] \,.\, \vee .\, p$

[*1·02] $\vdash : -p \,.\, \vee .\, p$

[*1·02] $\vdash : -p \,.\, \vee .\, p \vee p$

[*1·01] $\vdash : p \,.\, \supset .\, p \vee p.$

[2] *Principia Mathematica*, 2nd edition, p. 8.

[3] Nelson's solution of the problems of substitution has been disregarded here because it rests on the apparently unjustified assumption that the paradoxes of implication are true paradoxes and must be eliminated. This forces him to reject the perfectly valid proposition P . Q ∈ P, so that he may reject P . —P ∈ Q—a rejection which narrows considerably the significance of his system. The consequence to which he objects is eliminated by the application of rule 2 above.